WHY MATHS?

HOW MUCH MATHS SHOULD I DO FOR THE HSC OR AT UNIVERSITY?

A guide to students in year 10 or year 12 who are contemplating doing Mathematics for the HSC or at university

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You may not have a choice. What you really want to study may require a substantial amount of mathematics. Here are some of the areas of employment that need mathematics. I have ordered them roughly in decreasing order of the amount of mathematics they require.

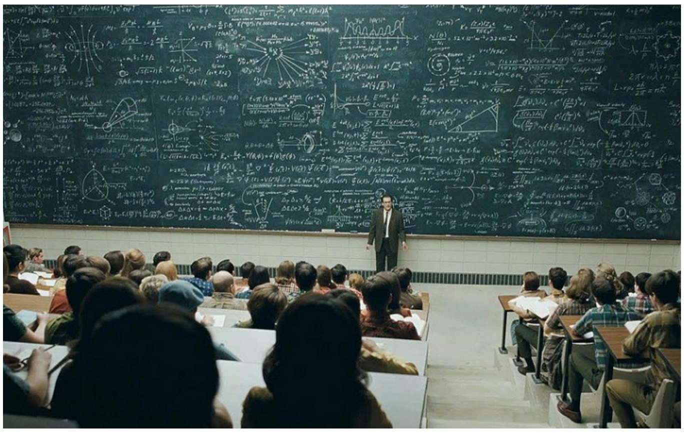
MATHEMATICS LECTURER

Of course, if you aspire to lecture mathematics at university, or do research into mathematics, you will clearly need to study a considerable amount of maths. You will need not just a university degree, majoring in mathematics – you will also need a PhD in maths, requiring three years, or more, of postgraduate study.

You should be aware that these days it's very difficult to get a permanent full-time job in a university maths department. Typically you would need to some post-doctoral research on fixed-term contracts before you can be successful in getting a tenured position. Very often this takes until one is in one's mid thirties, after having had a succession of three year contracts.

Sorry if this puts you off. You have not only got to be very good at mathematics, you also need a lot of imagination, you need to have the ability to be persistent and finally you need to be somewhat lucky. In my case I was somewhat good at mathematics, I have loads of imagination, I was persistent, but above all I was lucky. At the time I finished my PhD many new universities were springing up all around Australia. With no post-doctoral experience I applied for jobs at three universities and got offers from all three. In the current climate I wouldn't stand a chance at even one. You can only hope that the government decides to fund some new universities in the future.

But don't let me put you off. Mathematical research is an exciting occupation. You might think, as many do, that everything in mathematics was discovered a long time ago and that there is no further the need to do research in the subject. Nothing could be further from the truth. What you learn in high-school only brings you up to the 17th century. A university degree will contain very little before 1900. But Mathematics research has continued at an ever increasing pace since then. It's just that you don't get to hear about it. Once you enrol in a PhD you'll have to find something that is entirely new and this will require you to learn all that's known in a very narrow area.



The subject of mathematics continues to grow quickly in the 2020s. A number of years ago, before it went online, the publication *Mathematical Reviews* published a volume of brief abstracts of mathematical papers, containing entirely new material, every month. Typically it would describe a 20 to 50 page paper in just a paragraph or two. And yet it got to be the size of a small telephone book – every month!

But don't overlook the excitement of *teaching* mathematics at university. The talent of doing this well is often overlooked when it comes to promotion, but it is its own reward. Even though you may be teaching the same content year after year (because the new research that I talked about above might take fifty years or more before some of it reaches the university syllabus), the different students each year make it a refreshingly new experience each year.

With a first year course you might be lecturing to 500 students. Here you have to be somewhat of a showman to engage the interest of the students, pacing around in front of the theatre, using your hands, and sometimes shouting when you get to an exciting bit.

With a third year course it is much more intimate. With ten or twenty students you know each student by name and the lecture sometimes involves some discussion. Here there is the opportunity for students to ask questions and I really like it when I get a student who is prepared to ask a 'dumb question'.

A 'dumb question' is one that the other students think shows the ignorance of that student, but when it gets answered they discover that they didn't really understand the concept themselves – they only thought they did. Invariably the 'dumb' student ends up getting an A.

You may meet the terms Pure Mathematics and Applied Mathematics. Often they used to be studied in separate departments. Nowadays the terms have largely gone out of favour, but it's still useful to know what they mean.

In Pure Mathematics there is no connection with the real world. The mathematics is studied with no eye on possible applications. If the universe were to disappear overnight, pure mathematics would still remain – that is if there were minds to think about it.

It explores mathematical ideas purely for enjoyment, or to satisfy intellectual curiosity. It doesn't care whether there are any practical uses for it. In fact, in years gone by, some mathematicians have even boasted that some areas, like the theory of prime numbers, have absolutely no practical applications.

But don't get the idea that Pure Mathematics is of no earthly use. The crazy thing is that in many, many, cases an application has been found a hundred years or more after a discovery that was made just to satisfy mathematical curiosity. Somebody encounters a real world problem and lo, the tools for solving it have been sitting around for over a century.

For example, digital cryptography is the part of computer science that seeks to encode passwords or bank account numbers or military secrets. It was developed only about fifty years ago but it used the theory of prime numbers, and pure mathematical theorems about them, that were proved centuries ago!

Fourier Theory was developed in the nineteenth century as a purely intellectual exercise. Then a hundred years later along came Electronics and found Fourier Theory to be just what was needed.

Applied Mathematics takes an area of the real world and creates a mathematical model for it – that is a series of equations that describes it. Then, using mathematical methods it predicts outcomes. When the area of the real world is the physical world there's not a lot of difference between theoretical physics and applied mathematics. But mathematical models have been constructed in areas such as traffic congestion, geological forces, the weather, the economy, pandemics, and many more. I happen to be a Pure mathematician and so I will probably show my bias. I boast that only *Pure* Mathematics gets studied in Heaven!!

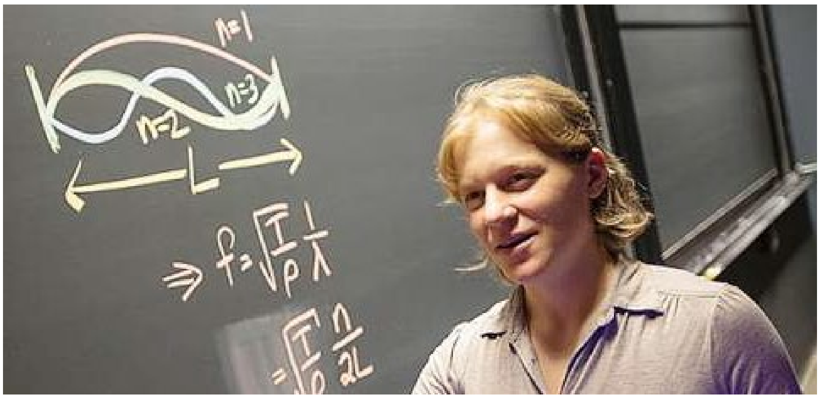
You should be aware that in Australia there are virtually no openings for mathematicians outside of the universities apart from the CSIRO which employs a few applied mathematicians.

ASTRONOMER

Astronomy and cosmology are mainly studied within physics departments in universities. But there are certain scientific organisations that employ astronomers. We're lucky in Australia because we're in the Southern Hemisphere and, with very few developed countries south of the equator there are more astronomers employed per 100,000 of population than in practically any other country. Astronomers usually need a major in mathematics at university in addition to a physics or astronomy degree.

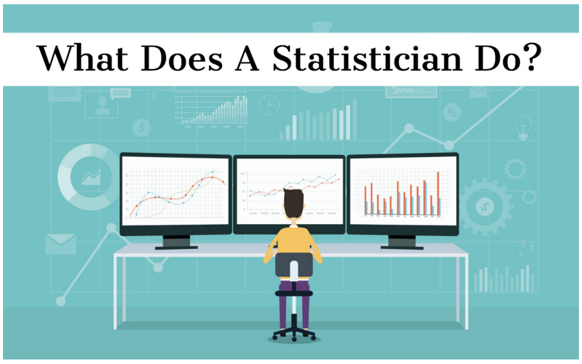
PHYSICIST

All areas of physics need a certain amount of mathematics. Experimental physicists perhaps don't need a huge amount but theoretical physicists certainly do. Certain areas, such as Quantum Physics, Particle Physics and Relativity, require a considerable amount of rather advanced mathematics.

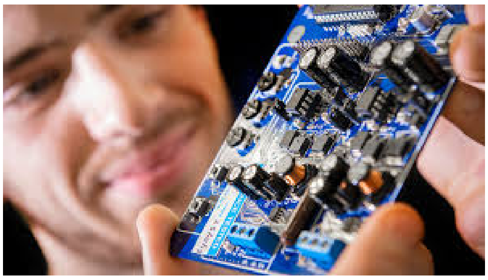


STATISTICIAN

There are two types of statistician – one who collects and analyses data and one who could be described as a mathematical statistician. The first type doesn't need a great amount of mathematics. But Mathematical Statistics is built firmly on mathematics. When I studied Mathematical Statistics at university, many years ago, we were not allowed to begin it until second year. We had to master first year Pure Mathematics first.



The main areas of mathematics that Mathematical Statistics uses are Calculus, Matrices and something called Measure Theory.



ELECTRONICS ENGINEER

Electronic engineers need a lot of Mathematics. Areas that are fundamental to the theoretical aspects of Electronics are Complex Numbers, Calculus and Differential Equations.

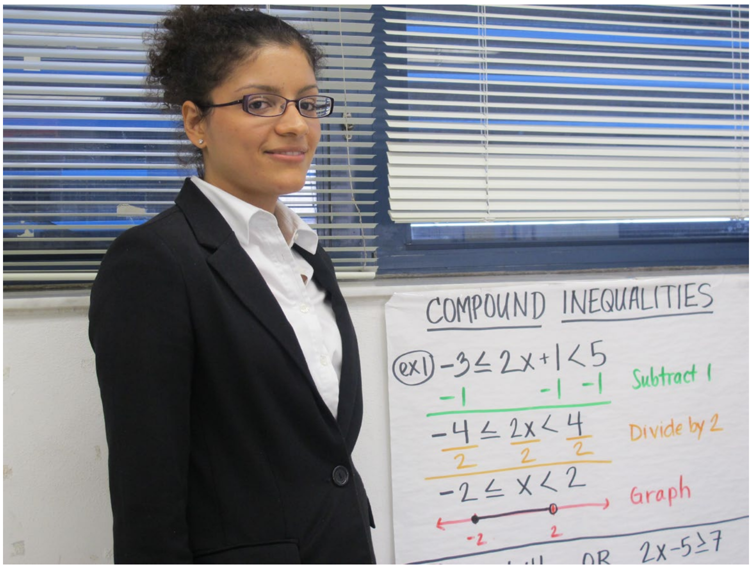
COMPUTER SCIENTIST

There are some branches of Computing Science that need very little mathematics and others that require a considerable amount. Relevant areas of mathematics are Algebra, Number Theory and Discrete Mathematics.

One of my sons did three years of university maths, along with his Computer Science. He has done very well as a top programmer and says that his mathematics was a complete waste of time because he's never used anything he learnt in maths! I have said to him that he has benefitted from the disciplined logical thinking that he learnt in his mathematics. I'm sure it has greatly assisted him in being a very sophisticated programmer – but he doesn't see it!

I have heard from recruiters in IT businesses that for many jobs in the industry they would prefer to employ a mathematics graduate with a certain amount of computer science than a computer science graduate with a bit of mathematics. It is a fact that a mathematics graduate can pick up whatever computer science they need on their own, while it is very difficult for someone with only a little mathematics to learn more on their own.

HIGH SCHOOL MATHS TEACHER



Someone who has completed the top level of maths at high school will know all they will need to teach. But if their mathematical training stops there they will make a pretty poor maths teacher. It is *so* important for a maths teacher to have seen a lot more mathematics than what they're required to teach.

One may enjoy high school mathematics but it's hard to get passionate about it. It is only in university that one gets to appreciate the whole picture of what mathematics is all about and to be set on fire by the subject. What is needed to be a good high school maths teacher is a real love for the subject, which only develops once you have seen the more fascinating parts of it.

FINANCIAL ANALYST

A lot of mathematics graduates end up employed as a financial analyst in banks. insurance companies or other financial institutions. This is because there are many more such jobs for mathematics graduates in these companies than in other parts of the economy. They use a small amount of what they learnt but more importantly the rigorous training they get by studying mathematics is important and valued. What they need to know about economics can be picked up easily on the job, though it doesn't hurt to know a little bit as part of one's degree.

ACTUARY

There's a misconception that actuaries stand at the top of the mathematics ladder. Not so! This belief has arisen because the HSC mark needed to do Actuarial Studies is much higher than for any other mathematics based degree.

Students doing Actuarial Studies at university are only required to do one semester of university mathematics. Now it is true that there's a lot of mathematical in all actuarial courses but it is mostly high school mathematics, or just a little more. But they have to work that fairly basic mathematics very hard and by no means could these courses be described as being easy.

But be warned – many students find actuarial studies boring. If the financial aspects fascinate you then you probably won't be bored. But if you drift into actuarial studies simply because you're good at mathematics then you may not like it.

You will learn very few interesting mathematical concepts by doing Actuarial Studies. Now I am not saying that nobody should study the subject. Many students do like it – and it does pay well. If the salary that you earn after graduation is important then by all means become an actuary. If you find out what actuaries do and find it fascinating then this is the degree for you. But if your motive for choosing Actuarial Studies is to explore mathematics to its uttermost depths, you will be disappointed.

Macquarie University, where I taught, was the first university in Australia to have an Actuarial Studies Program. This was great for the Mathematics Department in those days. Students with high maths marks were attracted to Macquarie for the Actuarial Studies and, although they did hardly any maths with us, we soon began to notice many disillusioned actuaries dropping out of Actuarial Studies and transferring to Mathematics. We got many excellent students this way. In more recent years the tendency has been to do a double degree – in Mathematics and Actuarial Studies. This gives the mathematically curious student the best of both worlds. They can graduate and get a job with a high salary and satisfy their mathematical curiosity as well.

ENGINEER

These days most of the mathematics that an engineer needs is done by software. A certain basic level, say with at least one year of mathematics, is useful so that you know what these packages are doing and when to apply them.

ACCOUNTANT

Accountants spend their whole lives doing Mathematics, but all they do with them is basic arithmetic, and their spreadsheets do that for them. So accountants had all the mathematical tools they need when they left Primary School. But despite the reputation of being boring I have found accountants on the whole quite interesting. However if they have any mathematical curiosity they won't get it from their employment.

It is quite common for me, when out with a group of friends, to be asked to check the bill and tell everyone what they owe. After all, they say, I'm a mathematician. My reply is invariably to say that I'm not an accountant. Accountants deal with numbers, while most branches of Higher Mathematics do not. Open a text book in some area of advanced mathematics and you'll hardly see any numbers. In many cases there are even very few equations too – just a lot of words with a few equations here and there.

WHAT DO I NEED TO BE A GOOD MATHEMATICIAN?

LOGICAL THINKING

For a start you need to be strong in logical thinking. This requires the discipline to follow through a chain of reasoning. The sort of logic that's used in the courts – where evidence is piled upon evidence until it tips the scales – is not relevant.

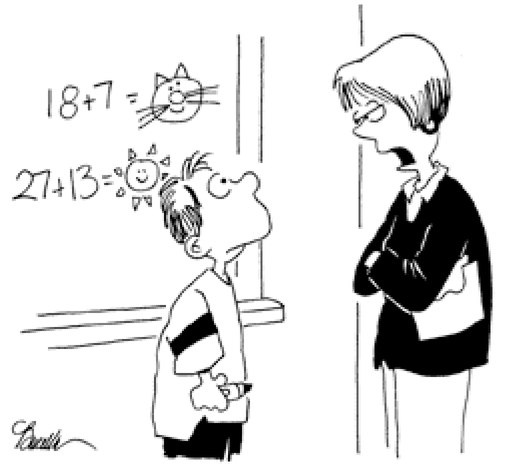
Then you need imagination. At Macquarie University we would often get students with very high maths marks from school who find university beyond them. At school you can do very well by being shown a method and doing hundreds of examples. A computer can be programmed to do this. At university, especially at the higher levels, original thinking is called for – being able to solve a problem where you haven't been shown the method, but which you can solve by piecing together facts that have been demonstrated in lectures.

IMAGINATION

Advanced mathematics is not just a case of clever problem solving. New branches of mathematics are very abstract and frequently hard problems are solved by inventing entirely new concepts to solve them.

The right way to solve a hard problem is to look at it in some new way so that it becomes an easy problem. Sometimes the path to the solution involves a detour into entirely different areas that seem to have nothing to do with the original problem. It requires flexible thinking to follow these detours.

Mathematics is often confused with Science. But they are quite different in their fundamental nature. The basic technique for establishing truth in Science is the Experimental Method, though some truths can be deduced by mathematical reasoning from facts which have been 'proved' experimentally. In Mathematics there are really no fundamental facts. Everything is proved by pure logic, starting with certain assumptions which cannot be proved.



"True, we have encouraged you to use your imagination, but not in Math."

Mathematics only exists in the human mind. Some people might claim that it exists in the mind of God, but we won't go into that. There's a famous saying that God invented the integers and man invented all the rest. But you could say that humans invented *all* of mathematics. Or you could say that God is the great Mathematician and that we merely rediscover what he has ordained, but I won't continue along these lines.

Mathematicians of old got hung up on whether certain mathematical entities really exist. When the square root of minus one was first contemplated it was called 'imaginary'. Yet these imaginary numbers proved extremely useful in solving real world problems. So they were allowed to be used but of course 'they don't really exist'. The modern perspective is that if something doesn't exist we invent it, provided such an invention doesn't lead to a contradiction.

The square root of minus one doesn't exist. Well, mathematicians said "let it exist" and they invented imaginary numbers and found them extremely useful.

Parallel lines don't meet. Mathematicians said "let them meet" – at an imaginary point (or as they're called an **ideal** point). In this way a new geometry was built called Projective Geometry. This is not just a piece of mathematical fantasy. Many theorems about ordinary Euclidean Geometry are much more easily proved using these ideal points.

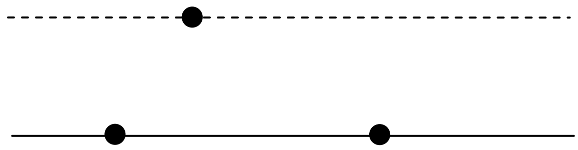
One of my mantras is that mathematician are the great storytellers of the scientific world. Nothing that we talk about actually exists, except in our imagination. We talk about infinitely long lines, with no end points – and these lines have zero width. Where do you find such things in the real world? They don't exist. We just made them up – using our imagination.

Have you ever seen a perfectly round circle? Or a point that has zero dimensions. All are useful fictions. Even numbers are just figments of our imaginations. You can't find them anywhere in the real world. We just made them up. Numbers only live in the human mind.

Mathematicians can happily prove theorems about 17 dimensional space. But a 17-dimensional world doesn't exist. Who cares! We can invent it and it lives in our minds. We don't attempt to visualise it, but that doesn't stop us proving theorems about it. Pure mathematics can be understood by disembodied angels.

A disembodied angel has nothing to do with religion. It is purely an imaginative concept dreamt up by one of my colleagues when teaching a Geometry course. A disembodied angel is a hypothetical being that is highly intelligent but has no concept of the spatial world.

Alan, my colleague, would send one of his students into another room with a walkie-talkie while another student went to the blackboard in the classroom – also with a walkie-talkie. Alan then asked the student at the board to describe a certain diagram to the disembodied angel in the other room. The disembodied angel had to pretend he or she had no concept of spatial concepts. To make it easy for you, who do have a concept of space, here is the diagram that was on the board.



“There are three points …”

“What’s a point?” the disembodied angel would ask.

There’s no way the student in the classroom could possibly describe a point to a disembodied angel. If you’ve never seen something resembling a point you could never know what it is. However one doesn’t really need to know the nature of a point to do abstract geometry.

“Well a point is a thing – never mind what it looks like. Do you know what ‘three’ means?”

“Of course. We call three things a ‘trinity’. So you’ve got three things, which you call points.”

“Then you’ve got this line.”

“What’s a line?”

“Well, it’s a different sort of thing.”

“Got it.”

“Now two of these points lie on this line, but the third one doesn’t.”

“I sort of understand – but what does it mean for a point to lie on the line?”

The student at the board starts to get frustrated. “Well, a point lies on the line if the line passes through the point.”

“Passes through?”

“Well just accept that there’s a relationship between points and lines called ‘lies on’.”

“Good, another undefined concept. So there are three points and a line with two points lying on the line and the third one not.”

“That’s it. Well in Euclidean Geometry there is a unique line passing through that third point …”

“Is ‘passing through’ a new undefined relationship?”

“No, it’s really the same as ‘lying on’, only the other way round. A line passes through a point if the point lies on the line. Now in Euclidean Geometry there’s a unique line … do you understand what ‘unique’ means?”

“Of course, it means ‘one and only’ – like God is unique.”

“Good. Then there’s a unique line through that third point that’s parallel to the original line.”

“Parallel?”

“Two lines are parallel if there’s a constant distance between them ..”

“Distance?”

“Let me put it another way. Two lines are parallel if they never meet.”

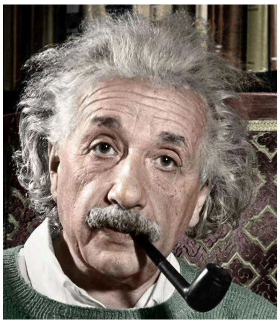
"Oh I understand 'meet'. When we angels meet we shake wings with one another."

"Not 'meet' in that sense exactly. Two lines are parallel if there is no point that lies on them both."

"Now you're talking. I understand perfectly. You have points and lines as undefined entities and 'lying on' as an undefined relation between a point and a line. So you have two points lying on a line and a third point that doesn't lie on this line. Then there's a unique line that passes through that third point such that there's no point lying on both lines."

"At last!"

WHO ARE THE THREE MOST INFLUENTIAL MATHEMATICIANS OF ALL TIME?



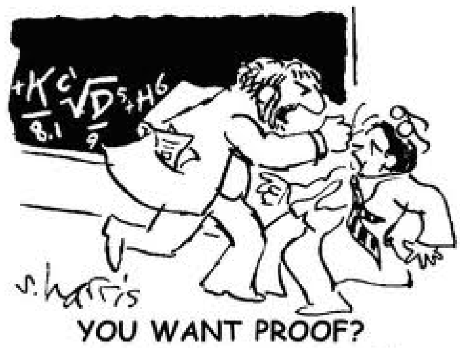
Einstein is often quoted as a great mathematician but he wasn't one at all. He was certainly one of the greatest theoretical physicists but he didn't discover anything new in mathematics. Mind you, he was no dunce when it came to advanced mathematics. He probably had a knowledge of mathematics comparable to a typical graduate student but, when the going got tough, he had mathematical colleagues who helped him out. But he would certainly make the top three theoretical physicists of all time. He had remarkable insight into physics and great imagination.

You may have heard of the Indian mathematician Ramunujan, the hero of the film *The Man Who Knew Infinity*. He, too, was certainly a great mathematician. But he didn't change the nature of mathematics as a whole.

Notice that I didn't say 'the most famous' because in my list you have probably only heard of two out of the three. My definition of 'influential' is to have changed the face of the whole of mathematics. Here are my three.

EUCLID

Euclid may have developed what we call Euclidean Geometry by himself or it may have been a team of mathematicians who worked under his name. I include Euclid not just because he was the founder of Euclidean Geometry but because he introduced the idea of a proof. He wrote a series of books under the title *Elements* in which he begins with a set of axioms. These he considered as being intuitively obvious, which they certainly seem to be. One of these is the so-called Parallel Axiom which states that given two points on a line and a third point not on that line, there's a unique line that passes through the third point and is parallel to the given line. You may remember the difficulty the student had in explaining this axiom to the disembodied angel.



But it is not for having developed what we now call Euclidean Geometry, that I include him in my list. It is because he introduced the concept of proof in developing that Geometry. Up to then Geometry was only an experimental science. Pythagoras' Theorem was known to be true because men had measured countless right-angled triangles and verified it. A physicist would have called it Pythagoras' Law. But Euclid *proved* that it was true. The concept of proof has since pervaded throughout the whole of Mathematics. Euclid really changed the face of Mathematics forever.

NEWTON

If I was writing this in German I would have replaced Newton by Leibniz. In the seventeenth century both Isaac Newton and Gottfried Leibniz invented what is known as Calculus independently – Newton in England and Leibniz in Germany. Newton was primarily an astronomer and he invented Calculus , or the Theory of Fluxions as he termed it, purely to help him study the motions of the planets. Leibniz was what is called a polymath – he was interested in many things. He was a philosopher, a mathematician, a psychologist, a lawyer and an engineer – though not, it seems, in astronomy. Leibniz's notation is the one that has been adopted since it's more appropriate for phenomena that don't involve motion.

More than half of what is taught, and probably half of mathematics research, is in the area of Calculus – or Analysis as it is called at the more advanced level. So Newton (and Leibniz) really changed the face of Mathematics.

GALOIS

Have you ever heard of Évariste Galois? Most probably not. That's probably because you don't do any mathematics at school that's associated with him. But, although he died in a duel at the age of 20, he left behind a legacy that has completely transformed the way we do mathematics today.

He was born on 25th October 1811 in France. He got bored at school and often failed his exams because he believed that what he had to learn was stupid. Instead, in his teens, he read advanced mathematics on his own. He failed to get into the prestigious École Polytechnique and instead enrolled in the much inferior École Normale.

At this time there was a lot of student unrest in Paris with opposition to King Louis Phillipe. He was in and out of jail for his political protests and he wrote much of his mathematics there.

A famous incident took place in May 1831 when some rioters, with whom Galois was aligned, were acquitted. There was a banquet held in their honour and it included many illustrious men, such as Alexander Dumas, the author of *The Three Musketeers*. During the festivities, Galois jumped up on the table with a dagger crying out "death to Louis Phillipe", followed by the words under his breath, "if he turn traitor". These last words were to cover himself, but not many heard them. A riot broke out, and Dumas jumped out of the window so that he could hide the fact that he attended such a traitorous gathering!

Galois was arrested the next day and was sentenced to six months. During this time he worked furiously on his research. He was released in April 1832. On 30th May 1832 he was challenged to a duel. It was supposedly over a woman but there is some evidence that it was politically motivated. He was shot in the stomach and died of peritonitis the following day. Before his death he had submitted a paper containing his research to the French Academy but it was lost. Many years later someone found it and realised how ground-breaking were his discoveries.

So what did he prove? His actual discovery has only somewhat minor interest. It is not for this that he makes it into my list of the top three. You will know from school that there's a quadratic equation formula which computes the solutions from the coefficients using the operations of addition, subtraction, multiplication and division and the extraction of a square root.

In 1515 a similar, though more complicated, formula was discovered for the cubic (involving powers of x up to x^3). Here the formula involved taking cube roots as well as square roots. In 1545 the quartic equation was solved in a similar way. So naturally the search was on to find a formula for the quintic, involving powers up to x^5. But Galois proved that no such formula can possibly exist.

Now although mathematicians are primarily involved in solving problems, one of the things they do is to prove that certain problems have no solution. Such a proof will then have the effect of preventing mathematicians from wasting their time in trying to solve the problem.

Of course that has never stopped the amateur mathematician from regarding this as a challenge. "So, they think it is impossible? I'll show them." Often they come up with what they consider is a solution and submit it to a mathematics department. They write back saying that there's a flaw in the solution and the amateur mathematician thinks that it's a conspiracy to protect the reputation of their colleague who claimed that it was impossible. But if a proof exists that something is impossible then no solution will ever be found.

However it's not this theorem about the quintic that makes Galois stand out above other mathematicians. His uniqueness consists in the methods that he invented to prove it. He invented the concept of a mathematical group and he developed quite a bit of Group Theory. When I was at Macquarie University I used to teach two whole third year courses that were based on his work: Group Theory and Galois Theory.

But even that is not what makes him really influential. Subsequent mathematicians changed the way they developed Group Theory by starting with four axioms. These define an abstract group. Then the whole theory is developed by starting with just those four axioms. Unlike Euclidean Geometry which attempted to develop a single mathematical structure, starting with the axioms as 'self-evident', the axioms of Group Theory are far from self-evident. They describe what is meant by a group and there's a whole wide world of examples of groups of all shapes and sizes. Groups have been connected with such odd phenomena as mail sorting and the marriage laws of Australian aborigines. More serious applications are found in Theoretical Physics and Crystallography.

This approach to mathematics – setting up a set of axioms to describe a certain type of mathematical structure and then proving theorems for these structures from these axioms – is the way most mathematical areas are now developed. So, although it wasn't Galois himself who set up these axioms, it was Galois' concept of a group that led to this axiomatic approach.

In teaching Galois Theory I have always had a lot of fun reminding my students, who are mostly about 20, that when he was their age he did this ground-breaking work!

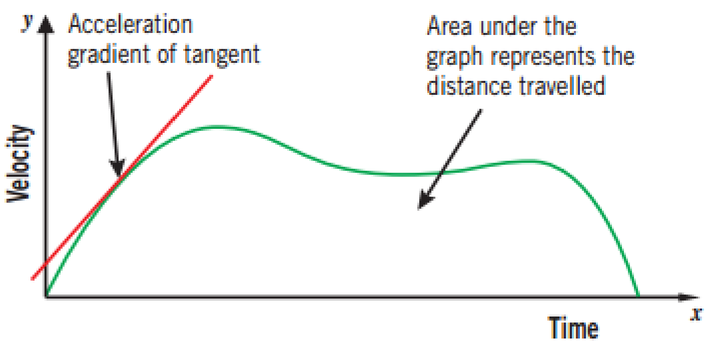
WHAT DOES MATHEMATICS LOOK LIKE AT THE ADVANCED LEVEL?

At school, and even at university, a student gets a pretty skewed picture of the mathematical world. It's just like someone who has lived all their lives in a small Australian country town and has a limited experience of the wonderful diversity of the world.

At school you're hear about Arithmetic, but hardly anything about Higher Arithmetic, or Number Theory as it is more usually called. You learn the basics of algebra, but nothing of Abstract Algebra. You learn of Euclidean Geometry, but nothing of non-Euclidean Geometry or Projective Geometry or Topology. You learn some Calculus, but not Higher Calculus which goes by the name of Analysis. Here is a very rough description of these and other areas.

CALCULUS

The simplest way to explain it is to think about graphs. A graph represents one variable plotted against another. The height of the graph, above the x-axis, represents the value of one variable given a certain value of the other. Height is just one aspect of a graph. The other two are **slope** and **area** under the graph. These represent other aspects of the connection between the two variables.



For example, if you were to plot the velocity, v, of a moving object against time, t, the slope would give you the acceleration at time t and the area under the graph, down to the t-axis, between time t_1 and time t_2, represents the distance travelled during this interval of time. There are many other situations where slope and area have very different interpretations.

If it is the graph of a mathematical function there are algebraic ways of calculating slope and area. Finding the slope for a given function is called **differentiating** the function and finding the area function is called **integrating** it. Obtaining a formula for the slope or the area certainly beats trying to draw tangents on an actual graph to estimate the slope, or counting squares to estimate the area under the curve.

Calculus looks mysterious because of its strange notation. If we have a graph of y against x, the derivative of y (or slope function) with respect to x is written $\frac{\mathrm{d}y}{\mathrm{d}x}$. You have to blame Leibniz for this strange notation – but it really is much better than Newton's. The integral, or area function would be written $\int y \, \mathrm{d}x$. You can now open any book on Calculus and while you may not understand the details, you'll recognise these notations, or variations of them, and you'll be able to say "I know what these are – that's the slope function and that's the area function".

DIFFERENTIAL EQUATIONS

This branch of Mathematics is really a part of Calculus but it is so important that it merits its own name in the Course Prospectus. Differential Equations are just equations that contain derivatives and solving the differential equation means finding a function that satisfies the equation. There are two branches: Ordinary Differential Equations and Partial Differential Equations, but I won't confuse you by explaining the difference.

The area of Differential Equations is one of the most useful parts of Mathematics and there are so many applications that I haven't space to list them.

ANALYSIS

This is the name given to the deeper end of calculus. You probably don't want to know any more at this stage of your life.

ABSTRACT ALGEBRA

Algebraically you have lived your whole life, perhaps not in Woop Woop, but perhaps in Sydney. Sydney is a great and important city but there's a lot more in the world than just Sydney.

The Arithmetic and Algebra that you learnt in school is about just one algebraic system – the system of numbers. In Kindergarten you learnt about whole numbers. In Primary School this was extended to include fractions. In High School you learn about decimal numbers and the number system had now grown to become what are called the Real Numbers. Later you might hear about Complex Numbers, which include the Real Numbers. It's just one big algebraic system.

Abstract Algebra asks what other possible algebraic systems there are. Now admittedly the most important one is the system of Complex Numbers, which include the Real numbers that are so familiar to us. But there are others, some of which are also useful.

The simplest type of algebraic system is the **group** – the system that Galois invented. There are other types but this will give you the flavour.

A **group** is any algebraic system where there's a way of combining any two things in the group (we will refrain from calling them numbers – instead we'll call them **elements**). We call the operating **multiplying** the two elements and the result is called their **product**.

In order for such an algebraic system to be called a group four axioms or properties, have to hold. If x and y are two elements we write their product as xy, just as in ordinary algebra. It is important not to bother asking what these elements are. They could be numbers, or they could be quite different things altogether. It's best to pretend that you're a disembodied angel and regard them purely as undefined objects. Likewise multiplication might just mean multiplying numbers in the usual way – or it could be something completely different. It's best to regard multiplication as an undefined operation. So we have a set of undefined things, called **elements**, and an undefined operation that we call **multiplication**.

The first axiom insists that xy must always be inside the group, the Closure Law. The second axiom says that $x(yz) = (xy)z$ whenever x, y and z are elements of the group. This is called the Associative Law and you probably remember being told about it being a law for ordinary algebra.

The third axiom says that there is a special element in the group which leaves every element unchanged when it multiplies that element. We write that special element by the familiar symbol 1, or sometimes I, and call it the **identity** of the group. But beware, it need not be the number one – it just behaves like it. In symbols we say that $1x = x$ and $x1 = x$ for all elements x in the group.

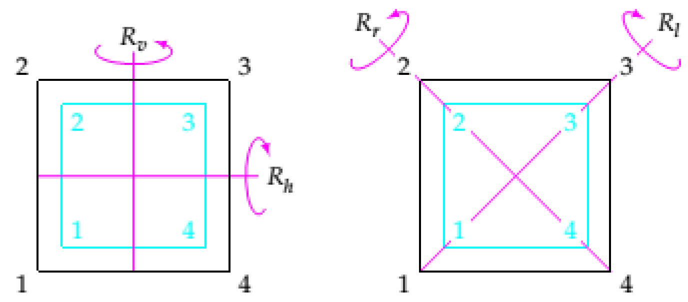
The last axiom says that for every element x there's an element, which we write as x^{-1}, such that $xx^{-1} = 1$ and $x^{-1}x = 1$. We use the familiar symbol x^{-1} by analogy with ordinary arithmetic but it usually doesn't mean $1/x$ in the usual sense.

This last axiom prohibits the system of Real Numbers from being a group if multiplication is interpreted as normal multiplication of numbers, because 0^{-1} doesn't exist. But if we exclude zero we *do* get a group. The product of two non-zero numbers is a non-zero number and the remaining three axioms clearly hold for multiplication of real numbers.

The system of non-zero integers, on the other hand, is not a group. Although 2^{-1} exists (we normally write it as ½) it is not an integer.

You will notice that we have omitted from the definition of a group the very familiar Commutative Law: $xy = yx$. That is we don't insist that the Commutative Law holds. Indeed in many of the most interesting groups it doesn't! So what is an example of such a group?

Take a square piece of cardboard and number the corners 1, 2, 3, 4 in clockwise order. Turn the square so that 1 and 2 are at the top and write the word START in the middle of the square. Now turn the square over and number them 4, 3, 2, 1 so that each corner has the same number on both sides. Of course these will now be 1, 2, 3, 4 in anti-clockwise order.



There are several operations that we can perform on this square that leave it occupying the same position in space, just with the corners being permuted in some way. You can rotate the square through 90 degrees or 180 degrees or 270 degrees, or even 360 degrees. Of course rotating it through 360 degrees is equivalent to a rotation through 0 degrees if you are just interested in the final positions of the corners. Then you can rotate the square through 180 degrees about any one of the four axes of symmetry: the horizontal axis, the vertical axis or either of the two diagonals.

The operation that we will consider as multiplication is the operation of doing one of these rotation followed by the same, or another, rotation. If R is the symbol we give to a 90 degree clockwise rotation about the centre then R^2 will denote a 180 degree rotation, R^3 will denote a 270 degree clockwise rotation (or a 90 degree anti-clockwise rotation) and R^4 will denote a 360 degree rotation.

Clearly a 0 degree rotation before or after any rotation will have no effect, so this will play the role of the identity, which we normally write as 1, but here we will write it as I. And since the 360 degree rotation is the same as a 0 degree rotation from the point of view of the final positions of the corners, we can write the equation $R^4 = I$.

Now let's denote the operation of flipping the square through 180 degrees about the horizontal axis by the symbol F. It's important to remember that these axes are fixed in space, not drawn on the square. So the horizontal axis will always be horizontal from your your perspective.

Clearly $F^2 = I$. Now, experiment with your cardboard square and convince yourself that $RF = FR^3$. That is if you start each time with the word START visible and the right way up, the positions of the corners will be the same no matter whether you perform RF or FR^3.

And clearly R^3 is the inverse of R so we can write $RF = FR^{-1}$. All the four group axioms hold so this is a group, but it is a non-commutative group.

If you experiment with your square you should be able to convince yourself that the 'flips' about the other axes of symmetry are RF, R^2F and R^3F. So the eight elements of this group are I, R, R^3, R^3, RF, R^2F and R^3F.

One of the more elementary theorems of Group Theory is that if the number of elements is a prime, or the square of a prime the group must satisfy the Commutative Law. This gives you some of the flavour of Group Theory and hence a little bit of the flavour of Abstract Algebra.

LINEAR ALGEBRA

The concrete version of Linear Algebra is the Theory of Matrices and Determinants. This is used a lot in Statistics and Engineering. A matrix is just a table of numbers. You can add and multiply matrices if the sizes are compatible (never mind what that means). To get some flavour of the subject let's stick to 2 × 2 matrices (2 rows and 2 columns). To add a couple of 2 × 2 matrices you just add corresponding components. Multiplication is more complicated:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1c_1 + b_1d_2 & a_1d_1 + b_1d_2 \\ a_2c_1 + b_2c_2 & a_2d_1 + b_2d_2 \end{pmatrix}.$$

Rather than remember this formula just examine the pattern. You run along a row of the first matrix and down a column of the second – multiplying and adding these products.

We've almost got a group here. It can be shown that this multiplication satisfies the Associative Law, and the identity is the matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The trouble is with inverses. Clearly $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has no inverse, but lots of other matrices fail to have inverses.

The **determinant** of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined to be the number $ad - bc$. It can be proved that if the determinant is zero the matrix has no inverse, but if the determinant is non-zero then it does. These matrices are called **invertible**. It can be shown that the invertible 2 × 2 matrices form a group. And, if you practice multiplying random matrices you'll see that it is a non-commutative group.

This is a tiny glimpse of the more concrete end of matrices and determinants. Linear Algebra proper begins with a set of axioms for something called Vector Spaces.

NUMBER THEORY

This basically just deals with the integers, or whole numbers. But it focuses on the concept of divisibility – one number dividing exactly into another. Numbers which only have two positive divisors, 1 and themselves, are called **prime** numbers. Notice that this excludes the number 1 which only has one positive divisor.

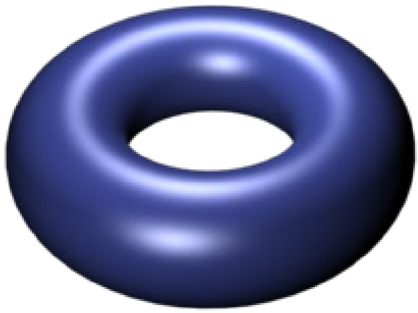
Number theory is largely about divisibility and prime numbers. It does include other things such as which numbers are the sum of two squares or what are the Pythagorean triples (whole numbers that can be the sides of a right-angled triangle).

It used to be thought that this was the most useless part of Mathematics but these days it underlies computer cryptography.

GEOMETRY AND TOPOLOGY

Not a lot of research is carried out these days into Euclidean Geometry or Projective Geometry – perhaps a little more into non-Euclidean Geometry. The big area of research is Topology. More and more applications are being found for this the bendy and stretchy geometry called Topology. It's a huge area, so let me tell you about just one small corner – the topology of surfaces.

Topology is mostly about connectedness. If you make a teacup out of plasticine you can squeeze the cup part so that it becomes part of the handle and stretch out the handle, so that you can make it into a doughnut shape. So, in topology a teacup is the same as a doughnut. If you tried to do this with a ball of plasticine you could only make a doughnut by making a hole, or rolling it into a tube and joining the ends. In topology you're not allowed to tear a hole, resulting in nearby points suddenly becoming far apart. Nor are you allowed to joint different parts so that some points which were far apart are suddenly brought very close. Well, you are allowed to do these things but you will end up with something that is topologically different.



So a **sphere**, the technical name for a ball, is different to a **torus**, the technical name for a doughnut. But a teacup is the same as a doughnut.

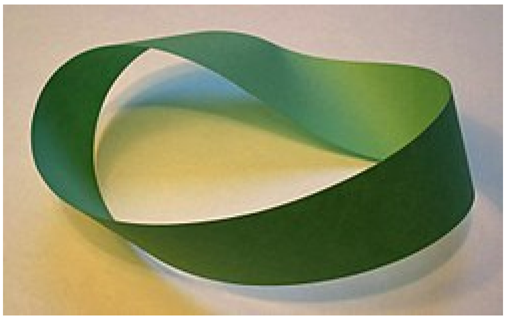
Doing this with actual plasticine would keep the volume constant, but in topology things are allowed to expand or contract and be considered not to have changed. Perhaps a better analogy is to think of balloons. An uninflated balloon is topologically the same as a disk, say the shape of a round piece of cardboard – just imagine stretching the balloon so that it lays flat and the neck of the balloon becomes the perimeter of the circle. You have to allow the plastic in the balloon to be infinitely stretchable! You can continue to stretch the flat circular piece of rubber so that it doubles in size. To a topologist it remains the same.

Now cut a hole in the middle of this circle. Cutting changes things topologically. The shape that you have is usually called an **annulus** – but a topologist would call it a **cylinder**. You can see why, by imagining that you stretch this annulus so that the inside circle is pulled out and enlarged to the same size as the outside circle. Pull these circles apart and you have … a cylinder. Topologically a cylinder is the same as an annulus.

Take a strip of paper. Topologically this is the same as a disk – a flat circle. You just have to stretch the strip sideways so that it becomes a square, and then stretch it at the corners so that the edge of the square becomes a circle.

Now take a real strip of paper and take the ends and join them up. Joining up changes things topologically and now you have a different surface, namely a cylinder. It's a pretty short cylinder if you stand it up on one of the circular edges but you can stretch it so that it looks taller and becomes like a tin can with both ends removed.

But now take another strip of paper. Give one end a half twist and join the two ends together. Joining changes things topologically and what we have is different to a cylinder – it's called a **Möbius Band**. You can see that it is different to the cylinder because a cylinder has an inside and an outside while the Möbius Band only has one side. A tiny insect on a Möbius Band can crawl over the whole surface while on a cylinder, assuming it can't negotiate the edge, it has to stay on one side.



Now the discussion so far might not seem like mathematics – why there have hardly been any numbers! It's not all like that. An advanced text on topology would start with a set of axioms which would cover more than just surfaces, and there would be formal proofs rather than excursions into one's imagination. But let's continue with this informal discussion.

Take your Möbius Band. The edge is one continuous circle – well it's not actually a real circle but it is, topologically. There aren't two circles as in a cylinder. The edge of a Möbius Band is just a single closed loop. Now take a flexible circular disk made of stretchy rubber (do this in your imagination). That also has a single edge that is a closed loop. Can you stitch these two closed loops together. If you try you'll find that it is impossible in 3-dimensional space. But it *can* be done in 4-dimensional space, and what you would end up would be a Projective Plane.

As I said there's a lot more to Topology than just this fun little corner of the subject. But it should be enough to show you that some parts of Mathematics look very different to what you learn at school.

KNOT THEORY

This is actually another fun part of Topology. If you take a piece of string and glue the ends together you'll get a circle. Then you can scrunch it up and wind it around itself, but as long as you don't cut it open it is still topologically a circle. But suppose, before you join the ends, you tie a knot and then join the ends. This is what Knot Theory studies. It has some applications into the study of DNA and the way the DNA gets knotted together.

Now one fundamental problem in Knot Theory is deciding when two knots are the same. If you can get from one to the other without cutting it open then topologically they are the same. If I were to give you two knots, that look quite different, you may be clever enough to manipulate one so that it looks the same as the other. Great! You will have proved that they are the same knot.

But what if you can't make one look like the other? Does this prove that they are different? They may, indeed, be different knots. On the other hand it may be that they are the same but you haven't been clever enough to get one to look like the other.

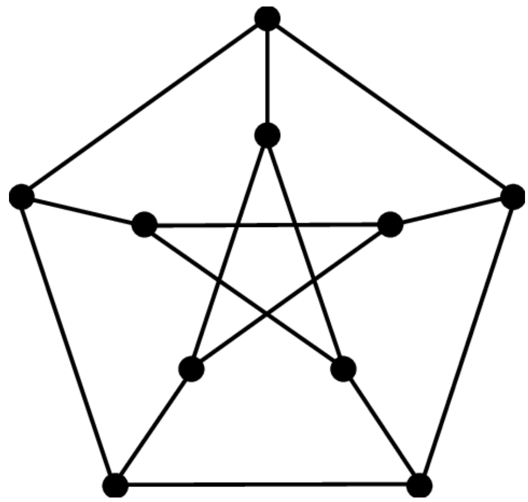
You can lay each knot flat and draw pictures of them, showing the unders and overs in the usual way at the crossings. The bit that goes under is drawn so that it appears to be broken. One thing you can learn to do in Knot Theory is to take such a diagram and do some arithmetic and come up with a number, called the Alexander Number. This number will always remain the same no matter how much you manipulate the knot.

So if you work out the Alexander Number for each of the two knots that I give you, and if one is 17 and the other is 19, you will have proved that they are indeed different knots. If, on the other hand both knots come out as 17 this doesn't prove that they are the same. It's a one-way test. It may mean that you have to try harder with your manipulation, or use a better test.

A variation on the Alexander Number is the Alexander Group. One can do certain calculations on the pictures of a knot and come up with a group. Another variation produces a polynomial. These are called 'invariants'. Different groups or different polynomials will prove that the knots are different. These invariants will often distinguish between knots that have the same Alexander Number.

GRAPH THEORY

Forget the graphs that you draw on graph paper. The graphs in Graph Theory are points and lines connecting certain points. The layout is not important and the lines can curl around and even cross over each other. As long as it is clear which points (we call them **vertices**) are joined to which. We call the joining lines **edges**.



Graphs are sometimes called networks and they can represent many different situations in the real world, such as traffic networks or relationships between people. Sometimes we add a direction to the edges, like one-way streets, and sometimes we attach distances to the edges which can represent actual distances but often something else.

One of the problems discussed in Graph Theory is the Travelling Salesman Problem, the problem of finding a path that visits all the vertices with the smallest total distance. Another is to investigate which graphs can be drawn on which surfaces in such a way the edges don't cross.

In particular, one can consider the so-called **complete graph** K_n with n vertices each connected to each other (no arrows). What is the largest n such that K_n can be drawn on a given surface?

On a plane, a sphere, or a cylinder you can place 4 vertices and join each to each other, without any edges crossing – but no more than 4. On a Möbius Band the maximum number of vertices you can do this with is 6. On a torus the maximum is 7.

A closely related area is map colouring. On a sheet of paper, no matter what map you draw (real or imaginary) you can always colour the regions, using 4 colours or less, so that adjacent regions (sharing a common border) have different colours.

On a sphere, or a cylinder this number is still 4. But on a torus there are maps that need as many as 7 colours. But every map on a torus can be coloured with 7 colours.

INFINITE SET THEORY

This is probably the weirdest branch of Mathematics. A **set** is just a collection of things. There doesn't need to be anything in common with the things in the set, which we call its elements. In *Alice in Wonderland* the Walrus sings of shoes and ships and sealing wax and cabbages and kings. One can form a set from one or more of each of these items.

In mathematics we usually stick to sets of mathematical objects, such as numbers. We also allow the elements of sets to also be sets. One way to describe a set is to list the elements, such as {1, 17, 32}. This only works for finite sets, unless one can spot a pattern. So {2, 4, 6, 8, …} might suggest the set of all even positive numbers. For infinite sets we usually have to resort to describing a property that describes the elements.

So {2, 3, 5, …} might suggest the set of all positive prime numbers but with just a few elements you can't be sure. Instead we would write $\{n \mid n \text{ is a positive prime number}\}$. You would read this as the set of all n such that n is a positive prime number. Or you could include the definition of prime and write it as:

$\{n \mid n \text{ is an integer and } n>1 \text{ and } n = ab \text{ for positive integers } a, b \text{ with } a \leq b \text{ implies that } a = 1\}$

About a hundred years ago there was an attempt to put all of mathematics on a sound logical foundation. The German mathematician Gottlob Frege attempted this and he sent his manuscript to Bertrand Russell to proof read it, in case there were any minor errors. Russell found just one *huge* error that forced Frege to withdraw his work from publication.

Frege assumed that for every property there is a set. For every adjective there is a corresponding noun. If P is a property, and Px means that x has that property, then Frege assumed that he could always talk about $\{x| Px\}$. Russell came up with the famous Russell Paradox which showed that there are certain properties where this leads to a contradiction.

Russell said, "what about the property x $\notin$ x?" Here $\in$ denotes 'belongs to', or is an element of' and $\notin$ denotes 'does not belong to'. If x was the set of all sets then clearly $x \in x$. The set of all sets is itself a set. But if x = {cabbage, king} then $x \notin x$ because the set x is neither a cabbage nor a king.

Now if S = $\{x \mid x \notin x\}$ Russell asked the question: which of these is true:

$$S \in S \text{ or } S \notin S$$

If S $\in$ S then it has the property that defines S, that is S $\notin$ S.
On the other hand if S $\notin$ S it has the property that defines S and so S is an element of S, that is S $\in$ S. If S $\in$ S is true then it is false, and if it is false then it is true – a contradiction!

This put mathematicians to work to fix this problem. One can form the set $\{x \mid Px\}$ only for certain properties P. The mathematicians Zermelo and Fraenkel came up with a set of axioms, called the ZF axioms. They said that a set is an undefined object, with one undefined relation of membership between them, subject to these axioms. One doesn't have to rely on one's intuition as to what set membership means.

A set could be an integer and $x \in y$ might mean that x divides y; except that this interpretation wouldn't satisfy all the axioms.

Based on the ZF axioms one can build up virtually the whole of mathematics in a rigorous and logical way. With this approach everything in mathematics is considered to be a set. The number 0 is defined to be the empty set { }, the set with no elements. It is allowed to be a set because of one of the ZF axioms. The ZF axioms permit us to consider {0} as a set and this we define to be the number 1. Continuing in this way the number 7 is defined to be: {0, 1, 2, 3, 4, 5, 6}. Again the ZF axioms allow this to be a set.

One can then define the arithmetic operations of addition and multiplication purely as set operations and again these are justified because of the ZF axioms. One can then prove the basic theorems of integer arithmetic in a formal and logical way.

The next thing is to define fractions, decimal numbers etc. as sets. This requires great ingenuity. From this one proves the basic theorems of Arithmetic. Points and lines in 2-dimensional space can be defined as sets and the axioms of Euclidean Geometry can now be proved. Hence all the theorems of Euclidean Geometry will be proved in the usual way. Indeed the entire edifice of mathematics can be constructed on the basis of the ZF axioms.

You may object that numbers and points are not sets. They are if we define them to be. Numbers and points are just made up things. There is nothing to stop us thinking of them as sets as long as these sets behave in the usual way as numbers or points.

And we don't ask whether the ZF axioms are true. We take them as a foundation in the same way as a religious creed. After all, you can't prove anything from nothing. You have to start by making certain basic assumptions. The Christians believes that the statements in the creed sound reasonable and so they accept them in faith. The mathematician believes that the axioms of set theory sound reasonable so these are accepted in faith. Please never belittle a Christian for believing things they can't prove. Mathematicians do it too. In fact nobody can possibly prove everything they believe because all proofs have to start with some basic assumptions.

Now one problem is that the ZF axioms have never been proved to be consistent and, because of their fundamental nature, they probably never will. That is to say, it is not impossible for someone in the future to come up with a contradiction arising out of the ZF axioms. That won't be a real disaster. Those who bother about the fundamentals will just modify the ZF axioms to avoid the problem while all the other mathematicians will just quietly go about their business.

One of the things one wants to do with sets is to count how many elements they contain. Of course for infinite sets we need infinite numbers – not just 'infinity'. In the late nineteenth century Georg Cantor showed that some infinite sets are actually bigger than others. With infinite sets you don't automatically make them bigger by throwing in extra elements.

If F = {1, 2, 3, …} and G = {0, 1, 2, 3, …} you might think that F is smaller than G because G has one extra element, but in fact they have the same size. We just pair off 1 from F with 0 from G, and 2 with 1 and so on. There is clearly a one to one correspondence here.

Cantor introduced the rather peculiar notation for the size of all the above sets: $\aleph_0$. The symbol $\aleph$ is the first letter of the Hebrew alphabet and the subscript 0 indicates that this is just the smallest infinite number.

The number of fractions is $\aleph_0$ because it is possible to write out a single infinite list that includes all fractions, both positive and negative. Clearly we can't just list them in ascending order because, for example, there's no next smallest fraction after 0. But when it comes to decimal numbers there's a quantum leap. There are $\aleph_1$ decimal numbers, and there are $\aleph_1$ complex numbers. And $\aleph_1$ is bigger than $\aleph_0$.

The question may occur to you "is $\aleph_1$ the very next infinite number or is there an infinite number between them?" The statement that $\aleph_1$ is the very next number after $\aleph_0$ is called the **Continuum Hypothesis**. Is the Continuum Hypothesis true or false? The answer is that we don't know. It's not that the question is still open and one day in the future it will be answered. No, **the matter is completely undecidable**.

I remember a student of mine saying, "what I like about mathematics is that you know where you stand – every statement is either true or false".

Well, I'm sorry. There's a third category: undecidable. You see it has been proved that on the basis of the ZF axioms it is logically impossible to prove that the Continuum Hypothesis is true. It has also been proved that it is impossible to prove that it is false. The question can never be answered because such an answer is a logical impossibility.

I happen to believe that it is true, on the grounds that if a specific example was ever found of an infinite number between $\aleph_0$ and $\aleph_1$ this would answer the question and so contradict the fact that such an answer is logically impossible. But this is not quite the same as proving it true. There could be such an intermediate number but we can't get access to it. It clearly seems a convenient belief that the Continuum Hypothesis is true. But because we can't prove it we can just add it to the ZF axioms.

You may be getting the impression that Infinite Set Theory is really weird. I once wrote a book called *Mathematics at the Edge of the Rational Universe*. It is logically sound mathematics but it gets close to the point where logic breaks down.

There's another famous undecidable statement in Infinite Set Theory. If I give you a number of boxes and show you that none of them is empty, could you select exactly one item from each box? Of course you could. But suppose there are infinitely many boxes and each one contains infinitely many items, could you still select one from each?

I'm sure you'd say that, apart from the practical difficulty of doing it in finite time (which we ignore), it is theoretically possible. Well you can believe that if you wish. Or you can deny it if that takes your fancy. It has been proven to be undecidable – something else you can add to your mathematician's creed if you wish.

The statement that such a choice is always possible is therefore an axiom, called the **Axiom of Choice**. You are free to add it as an axiom if you wish. Along with many other mathematicians I choose to do so because it makes the statements of certain theorems simpler if you allow it. Other mathematicians don't like it.

I must warn you that one of the consequences of the Axiom of Choice seems intuitively unbelievable, though this falls short of a proof that it is wrong. Suppose you take a solid sphere with radius of 10cm. If we assume the Axiom of Choice it is theoretically possible to cut this 10cm radius sphere into a certain number of pieces and, like in a 3D jigsaw, reassemble them to make two solid spheres, each with a radius of 10cm!

Wow! If we made the sphere out of gold this would be a great way to make one's fortune! But I said 'theoretically possible'. There is no way that one could do this in practice. A real solid sphere is made up of a finite number of atoms and there's no way such a process could double the number of atoms. The spheres we are talking about are homogeneous mathematical spheres and the pieces we would cut them into would be clouds of points – not something that any cutting instrument could possibly achieve. These 'pieces' are so nebulous that the concept of volume just doesn't apply.

I have no difficulty in allowing such a non intuitive phenomenon to be theoretically possible so I accept the Axiom of Choice. It gets thrown into the bag of mathematical things I can't prove but which I am happy to accept. Other mathematicians feel that they cannot accept it. It's their choice and it will never be possible for anyone to decide who is right.

You may worry that if an aeronautical engineer assumes the Axiom of Choice when designing an aeroplane it may fall out of the sky. Let me assure you that for any practical application you would never need to use the Axiom of Choice. The only practical benefit I can see of assuming the Axiom of Choice is that a few theorems can be stated more simply. So we save a tiny bit of ink! It is purely for aesthetic reasons that it suits me to 'believe' in the Axiom of Choice.

CATEGORY THEORY

This is a relatively recent branch of Mathematics, having only been around for about fifty years. Like Set Theory it attempts to be a foundation for the whole of Mathematics but it does it rather differently. It is currently a very active research area and has some applications in Computer Science.